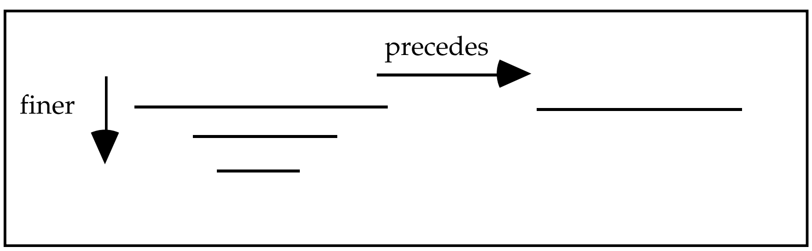
4. Interval theories

The axioms in this chapter all quantify over intervals of time. Intervals are not totally ordered, and a wide variety of relationships might hold between two intervals even if we assume, as I usually will in this section, that time is linearly ordered. Allen (1984) lists thirteen relations; equality and the following six plus their inverses (ie with arguments reversed) <u>precedes,</u> <u>meets,</u> <u>overlaps,</u> <u>starts,</u> <u>finishes</u> and <u>during.</u> This is often taken to be a standard set, but we will also consider theories which use subsets of these and which use different sets of relations. Any relation between intervals (on the line) can be defined in terms of these six, but smaller sets also suffice to define them all. We will give a theory expressed entirely in terms of <u>meets</u> within which all the relations can be defined, for example. These thirteen relations are all the relations which completely specify the relative orderings of the endpoints of the intervals. This suggests a way to define them within the timepoint theories, and we also present a theory based on this intuition.

The distinction between dense and discrete time also runs through these theories. Most of them can be extended by axioms which restrict to one or the other case, but they are also often consistent with the idea that time is dense in some places and discrete in others.

4.1 Intervals as information

An interval might be thought of as giving partial information about the location of a point: as an approximation to a point, in a sense. Two intervals might be sufficiently distinct that they establish the relative ordering of the points they contain: following Allen, we will call this relation `precedes` (although we think of it as more like the disjunction of his *precedes* and *meets* relations.) On the other hand, one interval might surround the same point as another but more precisely; let us call this `finer`. Both are orderings, but `finer` is partial and `precedes` is total.



The following axioms constitute a theory called APPROXIMATE-POINT, since it treats intervals as approximations to points.

```
;;AP-Syntax
(chained precedes finer)

;;AP-Trans-1.
(forall (?x ?y ?z) (=> (precedes ?x ?y ?z)
                        (precedes ?x ?z)))

;;AP-Trans-2
(forall (?x ?y ?z) (=> (finer ?x ?y ?z) (finer ?x ?z)))

;;AP-reflex
```

```
(forall (?x) (and (not (precedes ?x ?x)) (finer ?x ?x)))

;;AP-finer-asym
(forall (?x ?y)(=> (finer ?x ?y ?x)
                   (= ?x ?y)))
```

If two intervals are not clearly separate, then they must somehow intersect. That is, there must be a shorter interval contained in them both. This leads to the axiom corresponding to the total-order assumption for points:

```
(forall(?x ?y)(or (precedes  ?x ?y)
                  (precedes  ?y ?x)
                  (exists (?z)(and  (finer  ?z ?x)
                                    (finer  ?z ?y)))
      ))
```

It is easier to state this using an intermediate concept of 'not clearly distinguishable from', or `ncdf`:

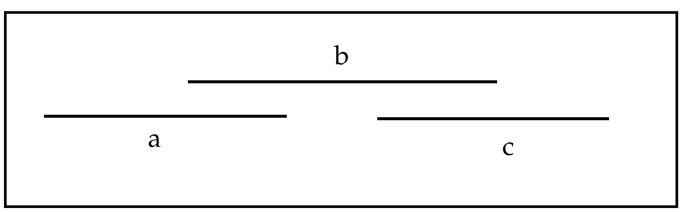
```
;;AP-ncdf
(defrelation ncdf (?x ?y) :=
  (exists (?z) (and (finer ?z ?x) (finer ?z ?y))))
```

`ncdf` plays the role here that equality does in the linear-point theory (which might be called an exact-point theory):

```
;;AP-orthogonal-1
 (forall (?x ?y)
         (or (ncdf ?x ?y) (precedes ?x ?y) (precedes ?y ?x)))
```

Comparing this with `before-LP-linear` shows how intervals can be interpreted naturally as approximations to points.

The relation `ncdf` is not transitive, since `(ncdf a b)` and `(ncdf b c)` are consistent with `(precedes a c)`:



However, under these circumstances there must be two subintervals `e` and `f` of `b` such that `(precedes e f)`, ie the overlapping interval can be separated into two subintervals which are order-distinguishable.

We need a few more axioms which establish the connection between `precedes` and `finer` :

```
;;AP-orthogonal-2
(forall (?x ?y) (not (and (finer ?x ?y) (precedes ?x ?y))))

;;AP-separation
(forall (?x ?y ?z) (=>
                    (and  (finer ?x ?y)
```

```
                              (precedes ?y ?z))
                       (precedes ?x ?z) )
        )
```

It now follows that the disjunction in AP-orthogonal is exclusive: for if `(finer ?z ?x)` and `(precedes ?x ?y)`, then `(precedes ?z ?y)` by separation; but `(finer ?z ?y)`, contradicting the second orthogonality condition.

Notice that of Allen's relations, precedes, overlaps , and during are the only ones which do not specify that two endpoints are identical, so we might expect them to correspond to these relations. Following the claim that we can only have approximate information, let us say that meets, start, and finish are always false. Then we could read `ncdf` as meaning the disjunction of equality and the relations {`o,d,di,oi`}.

The infinite extension of the timeline can be asserted in the obvious way:

```
;;AP-infinite-past
(forall (?x) (exists (?y) (precedes ?y ?x)))

;;AP-infinite-future
(forall (?x) (exists (?y) (precedes ?x ?y)))
```

However, this still does not guarantee that any large intervals exist. For example, the set of all rational intervals less than unit length would satisfy the axioms so far. One way to assert this is to claim explicitly that there is an interval which is less precise than any two intervals:

```
;;AP-large
(forall (?x ?y) (exists (?z) (and (finer ?x ?z) (finer ?y ?z))))
```

Models

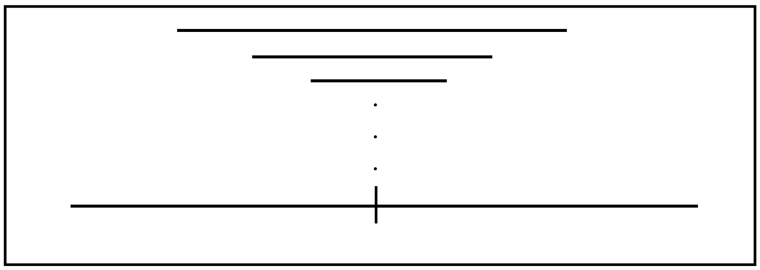
> This theory, in spite of its claims to infinite extension, can be straightforwardly interpreted as being entirely about the open subintervals of the unit real interval. The axioms guarantee that there there will always be a future, but they say nothing about how long that future may last. This can only be achieved by talking about the durations of intervals, as in section 6.

Density, which in a sense is the opposite of AP-large, is also straightforward:

```
;;AP-dense
(forall (?x) (exists (?y) (and (finer ?y ?x) (not (finer ?x ?y)))))
```

Although this way to describe density is fairly standard, its practical utility is not so obvious. The axiom rules out non-dense models by insisting that finer and finer distinctions must be possible. If we skolemise this and think of it computationally, it can be interpreted as saying that finer *measurements* are always possible. In practice this is not usually the case, and there is a definite limit to the precision with which measurements can be made, a 'grain size' past which it is not possible to discriminate separate points; or, equivalently, a size below which intervals seem like points. We could just say that the world is therefore 'really' discrete, as is usually done in temporal databases, where times are usually reckoned as integer counts of a clock-tick (Snodgrass et al 1994). But this is unsatisfactory for several reasons. We would like to be able to work with the assumption of density up to a point, as it were; and second, discreteness, as it is usually described, actually is too rigid. For example, it is quite possible for two clocks, both at our limits of discernment, to be beating out of phase with each other; but this is impossible if *time itself* is reckoned to be discrete.)

Anyway, its about the only apple in the shop, so we will adopt it as the density axiom. Given this, the Allen relation of meets can be defined using the idea that the meeting place can be contained in finer and finer intervals. If two intervals meet, and a third overlaps the meetingpoint, then if the line is dense it is always possible to find a subinterval which also overlaps that point and hence is *notclearlydistinguishablefrom* both the touching intervals:



```
;;meets-AP-dense
(defrelation meets (?x ?y) :=
             (and (precedes ?x ?y)
                  (forall (?z)
                          (=> (ncdf ?x ?z ?y)
                              (exists (?u)
                                      (and (finer ?u ?z)
                                           (ncdf ?x ?u ?y)))))))
```

Discreteness can be defined by following the intuitions behind those in the point axioms. First define a notion of an atomic clock-tick:

```
;;moment-ADP
(defrelation moment (?x) :=
             (not (exists (?y) (and (finer ?y ?x)
                                    (not (= ?x ?y))) )))
```

or equivalently

```
(defrelation moment (?x) := (forall (?y)(=>(finer  ?y ?x)
                                           (= ?x ?y)) ))
```

Discreteness could now be stated by the claim that all intervals have a next moment and a previous moment, following the definitions in the linear-point case; if, that is, we could express 'meets'. To simply assert that there is a next *interval*, which works for point orderings, is not enough here, as this is true even on the rational line. The definition given for the dense case does not work here, but a simpler alternative is available:

```
;;meets-ADP-discrete
(defrelation meets (?x ?y) :=
             (and (precedes ?x ?y)
                  (not (exists (?z) (precedes ?x ?z ?y))) ))
```

And now we can state discreteness by

```
;;ADP-discrete
(forall (?x)
        (and (exists (?y) (and (meets ?x ?y) (moment ?y)))
             (exists (?y) (and (meets ?y ?x) (moment ?y)))))
```

Models

Open intervals on the rational line **Q** or the real line **R** are probably the most intuitive models. The axioms do not distinguish between open and closed intervals, and can be interpreted in either way. For example, consider the universe of all open rational intervals and define `precedes` in the obvious way by (a,b) `precedes` (c,d) just when b is less than or equal to c, and `finer` as the weak subinterval relation. The same kind of nonstandard orderings ($\mathbf{Q} \uparrow \mathbf{Q}$, ie **Q** copies of **Q**, etc.) are also models here, and for the same reason. They now take on some extra style, since there are now intervals which span entire copies of the rational line. In the nonstandard models there are also special nonstandard intervals which do not have endpoints, such as the interval in $\mathbf{Q} \uparrow \mathbf{Q}$ consisting of all rationals <A,b> for some fixed A.

AP does not requre *all* set-theoretic intervals to be in the universe, however. For example, it is satisfied on the integers. Indeed, a model of AP can be made from any model of the point theories by selecting any infinite subset E of points, and letting the universe of intervals be all pairs of points <a,b> with a,b $\in$ E and a before b. Then two intervals meet just when they share an endpoint, and moments exist wherever E is not dense.

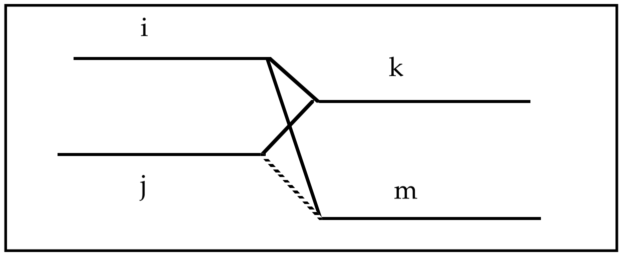
4.2 Intervals meeting in the glass continuum

Returning now to the idea of the continuum as consisting of intervals which meet *at* points, we assume only a single relation, `meets`, which is chained but not transitive. This theories can be regarded as axiomatisations of the intuition of the glass continuum.

First we specify that when two timeintervals meet defines a definite temporal location. This axiom 'ties together' the various ways a meetingplace of several timeintervals could be specified.

```
;;IM-syntax
(predicative timeinterval)

;;meets-IM-place
(forall (?i ?j ?k ?m)
        (=> (and (meets ?i ?k) (meets ?j ?k) (meets ?i ?m))
            (meets ?j ?m)  ))
```



Time is infinite:

```
;;meets-IM-infinity
(forall (?i) (exists (?j ?k) (meets ?j ?i ?k)))
```

Until now this has been an essentially arbitrary assumption, but here it takes on extra importance since proofs often need those extra intervals to nail down the position of a meeting-point. Notice however that being infinite, as noted earlier, does not guarantee infinite duration unless further assumptions are made.

That time is ordered can be stated by following the intuition that a meetingplace is an endpoint, and insisting that these points be ordered:

```
;;meets-IM-total-order
(forall (?i ?j ?k ?m)
        (=> (and (meets ?i ?j) (meets ?k ?l))
            (or (meets ?i ?l)
                (exists (?n) (or (meets ?i ?n ?l)
                                 (meets ?k ?n ?j) ))
                )))
```

If the 'or' here were exclusive disjunction this would suffice, but we need to add the following to prevent time becoming circular:

```
;;meets-IM-line
(forall (?i ?j) (not (meets ?i ?j ?i)))
```

The transitivity of the timeorder corresponds to the assumption that two meeting intervals form a single larger interval:

```
(forall (?i ?j ?k ?m)(=> (meets ?i ?j ?k ?m)
                               (exists (?n)(meets ?i ?n ?m)) ))
```

It is often convenient to replace this by its skolem form, using an explicit addition function on intervals:

```
(deffunction plus
(forall (?i ?j ?k ?m) (<=> (meets ?i ?j ?k ?m)
                            (meets ?i (plus ?j ?k) ?m)
                           ))
)
```

(This is not a logical definition as it does not specify a value for `(plus a b)` when `a` and `b` do not meet.)

This provides a neat way to define the notion of a `moment`:

```
;;moment-IM
(defrelation moment (?i) :=
  (forall (?j ?k) (not (= ?i (plus ?j ?k)))
   ))
```

These axioms constitute a small but surprisingly powerful theory within which all other binary temporal relations on intervals can be defined.

Density and discreteness can be defined in obvious ways:

```
;;IM-dense
(forall (?i) (not (moment ?i)))

;;IM-discrete
(forall (?i) (and
              (exists (?j)
                      (and (meets ?i ?j) (moment ?j)))
              (exists (?j)
                      (and (meets ?j ?i) (moment ?j)))
              ))
```

Models

These axioms have models which reflect several of the intuitions which motivate the glass continuum, all of which can be regarded as in some sense "standard" models, although they are not mutually consistent. First consider the dense case.

One model interprets intervals as open connected subsets of the rational line **Q** (ie open intervals in the conventional mathematical sense of 'interval'), asserts that (a,b) `meets` (c,d) just when b=c, and defines `plus` as the interior of the set-theoretic union of the closures of the intervals being added together. Notice then that the sum of two intervals contains the point where they meet. In general, if more complex operations are defined, they are modelled by always taking the interior of the corresponding set-theoretic operation applied to the closures of the arguments. The intersection of two meeting intervals is always the interior of a single point, ie empty; so two meeting intervals are disjoint.

This model amounts to a response to the intuitive puzzle discussed earlier which says that the point of division is in *neither* half of the split interval, but just somehow vanishes when they are considered seperately.

Alternatively, a dual model is provided by closed connected subsets of **Q**, with subsets meeting just when they share an endpoint, and the sum of two intervals being defined to be the closure of the set-theoretic union of their interiors. This means that two intervals which meet share the meeting point, but their interval-overlap is the closure of the interior of a singleton, ie the empty set. Thus there is no shared interval to play the role of `?n` in the definition of `overlap`.

This answers the puzzle by saying that the point of division is in *both* halves, but still insists that the two halves have an empty intersection. (The point might be thought of as identifying the surfaces of the two pieces of glass filament, which were in the same place before the break.)

A third way to model this theory thinks of every point on the line as having two halves, one facing to the past and one to the future. Suppose we have two distinguished copies of **Q**, and write <a| for a member of one copy and |a> for the other: call these the past and future "halves" of the timepoint <a|a>. An interval denotes a pair <a|b> where a is less than b; <a|b> `meets` <b|c>. The meeting point <b|b> is not an interval and is therefore "invisible"to the theory. While this may seem unnecessarily baroque, it supports the common naive intuition, when faced with the midpoint puzzle, that the midpoint itself must somehow be divided into two equal parts.

Note that these are all different from the standard model for the point continuum, in which a closed interval can only `meet` an open interval. They all refuse to distinguish between two kinds of interval, for example.

These theories can also be interpreted as referring to the point continuum, however. For example, one model is provided by the set of semiclosed rational intervals (a,b] where (a,b] `meets` (b,c]. Here the meeting-point clearly belongs to the first interval, and a timeinterval is interpreted directly as a mathematical interval.

Nonstandard models analogous to the nonstandard models of the point ordering theories also exist, of course. In fact, any model of a point-ordering theory can be extended to a model of the corresponding (discrete or dense) interval-meeting theory by defining intervals, and the `meets` relation, in any one of the three ways just described.

Analogous models of the discrete theory can be constructed on the integers, but here simpler models are possible. One defines `meets` as the relation between [n,m] and [m+1,k]. Here integers label moments [n,n], and meeting-points have to be thought of as <u>between</u> integers. Another model consists of the integer intervals [n,m] with $n < m$ and defines `meets` as the relation between [n,m] and [m,k]; here integers label points and a moment is an interval [n, n+1]. A nonstandard model here might be provided by **Q** copies of **N**, which, as Van Benthem points out, is "locally" discrete even though having a dense global structure.

Models of the basic theory can consist of moments inserted into other regions of density. Almost any combination is possible: in particular, nonstandard orderings such as **Q** followed by **N**, or **Q** with several copies of **N** inserted into it.

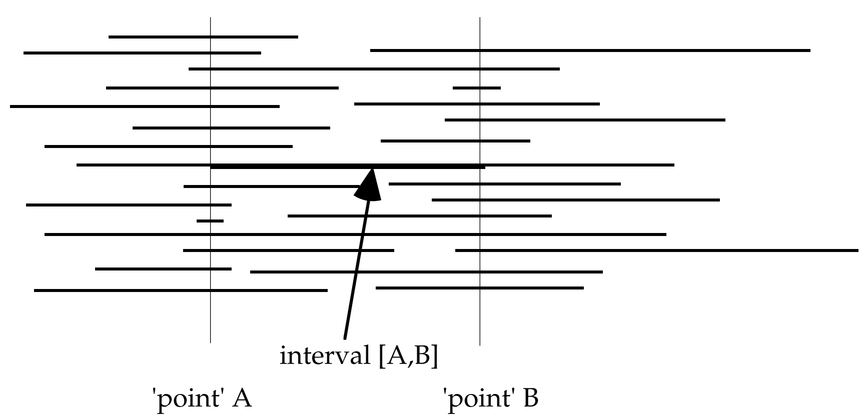
One class is of special interest, which I will call *point-moment* models. Consider a continuous interpretation such as **Q** or **R** with some selected subset M of isolated points (that is, for any $p,q \in M$, there must be a point r with $p < r < q$), and define an interval as a pair <a,a> where $a \in M$, or <a,b> when $a<b$; and `meets` be true of <a,b> and <b,b> (and of <b,b> and <b,c>) when $b \in M$, but of <a,b> and <b,c> otherwise. M can be any subset of isolated points ; it can be finite or infinite, and its members can be arbitrarily close to one another. Moments are identified by the members of M.

In such a model, a moment is being interpreted as a point, which is possible precisely because a moment has no internal structure. Point-moment models mix together some aspects of density and some of discreteness, so not surprisingly they are not models of either of the extended theories. (Not of density because moments exist, and not of discreteness because intervals with endpoints not in M – which must exist since M is isolated – do not satisfy the discreteness axiom.) These models are discussed further in section 5.3.1.

Constructing points in models of the intervals.

Since this theory is so firmly based on the concept of an interval, it may be surprising that any model of it contains things that can be regarded as timepoints. This is less trivial than it sounds, since in many models mathematical points (eg rational numbers) are not suitable interpretations of timepoints, especially in models based on the glass continuum. The interest of this result is that these theories are sometimes claimed to be 'point-free', in the sense that they are described purely in terms of intervals *sui generis*, rather than considering intervals as being defined by endpoints or consisting of sets of points. But such a claim has to taken with caution, since it is always *possible* to interpret this theory as talking about intervals constructed from points.

Suppose M is any model of these axioms with universe (of intervals) U, and consider pairs <B,A> (for 'Before' and 'After') of subsets of U with the property that every member of B `meets` every member of A; call such a pair a focus. The idea here is that a focus locates the meeting-point. Suppose a focus is *maximal* when the sets B, A are as large as possible, so that B contains every interval in U that meets a member of A, and vice versa. Then maximal foci (called 'nests' in Hayes & Allen 1987) can serve as points. The ordering relation `before` of section 3.1 can be defined thus: <B1,A1> is before <B2,A2> just when $A1 \cap B2$ is nonempty. The interval in the intersection is the interval between the points:



The function `beginof` from intervals to points is defined by: (beginof i) is the point <B,A> with $i \in A$; similarly for endof but $i \in B$. That these are unique follows from the axioms and the assumption of maximality. It is now straightforward to show that with these interpretations, all the axioms of the linear point theory LP and all the interval-endpoint definitions given below are satisfied. Thus, any model of the interval-meeting axioms can be interpreted also a model of LP . This works equally well for "nonstandard" models.

This filter construction, originally due to A. N. Whitehead, may seem unintuitive: a point is very small, but these maximal foci are very large. But when one considers that the role of the focus is to isolate a point as precisely as possible, then the idea that it might take an infinite amount of information to isolate something infinitely small makes the filter seem somewhat more compelling.

4.3 Thirteen Relations

Probably the most familiar temporal theory is often described as being based on the thirteen relations described earlier. In this section I will use the compact notation used by Allen (1984) and extended by Freska(1992) where each relation is denoted by its intial letter , followed by `'i'` to indicate the inverse.

The thirteen relations are an exhaustive and mutually exclusive set:

```
;;TT-list
(forall (?x ?y)
    (=> (timeinterval ?x ?y)
        (xor (p ?x ?y)
             (o ?x ?y)
             (m ?x ?y)
             (s ?x ?y)
             (d ?x ?y)
             (f ?x ?y)
             (= ?x ?y)
             (fi ?x ?y)
             (di ?x ?y)
             (si ?x ?y)
             (mi ?x ?y)
             (oi ?x ?y)
             (pi ?x ?y) )
     ))
```

Notice the use of exclusive-or here. This axiom can be written rather more tediously without it.

```
;;TT-inverses
(forall (?x ?y)
        (and (<=> (p ?x ?y) (pi ?y ?x))
             (<=> (m ?x ?y) (mi ?y ?x))
             (<=> (o ?x ?y) (oi ?y ?x))
             (<=> (s ?x ?y) (si ?y ?x))
             (<=> (d ?x ?y) (di ?y ?x))
             (<=> (f ?x ?y) (fi ?y ?x)) )
        )
```

The various relations between these are often, following Allen, summed up in a transitivity table. This 13 x 13 array gives the set of possible relations that can hold between x and z when one relation holds between x and y and the second between y and z. We summarise the table here as a variadic relation `ttable` on relations. The first two are the table coordinates, the rest are the possible entries at that point. This is expressed in the following second-order assertions, written here in pseudo-KIF notation:

```
4.3.3   (forall (?r ?s ?e @l)(<=>
                (ttable ?r ?s ?e @l)
                (forall (?x ?y ?z)(=>(and (?r ?x ?y)(?s ?y ?z))
                               (disjoin ?x ?z ?e @l) )) ))

4.3.4   (defrelation disjoin (?x ?z ?e @l) :=
                    (or   (?e ?x ?y)
                          (disjoin ?x ?y @l))
        )
```

Unlike the Dedekind axiom discussed earlier, however, these are not essentially second-order quantifiers. These 'second-order' variables are really only schematic placeholders for first-order relations: indeed, for our purposes, they need only refer to the twenty-six realtions in our current list. These axioms can therefore be quite reasonably transcribed into KIF by using the 'holds' relation between a relation and its arguments:

```
;;table-TT
(defrelation ttable (?r ?s ?e @l) :=
         (forall (?x ?y ?z) (=> (and (holds ?r ?x ?y)
                                     (holds ?s ?y ?z))
                                (disjoin ?x ?z ?e @l) )) ))

;;disjoin-TT
(defrelation disjoin (?x ?z ?e @l) :=
             (or (holds ?e ?x ?y)
                 (disjoin ?x ?y @l))))
```

The table is then summarised in the following rather long conjunction:

```
(and  (ttable p p p)
      (ttable p m p)
      (ttable p o p)
      (ttable p fi p)
      (ttable p di p)
      (ttable p si p)
      (ttable p s p)
      (ttable m p p)
      (ttable m m p)
      (ttable m o p)
      (ttable m fi p)
      (ttable m di p)
      (ttable o p p)
      (ttable o m p)
      (ttable fi p p)
      (ttable p d p m o s d)
      (ttable p d p m o s d)
      (ttable p f p m o s d)
      (ttable p oi p m o s d)
      (ttable p mi p m o s d)
      (ttable m si m)
      (ttable m s m)
      (ttable m d o s d)
      (ttable m f o s d)
```

```
(ttable m oi o s d)
(ttable o d o s d)
(ttable o d o s d)
(ttable fi d o s d)
(ttable m mi fi = f)
(ttable o p p)
(ttable o m p)
(ttable o o p m o)
(ttable o fi p m o)
(ttable o di p o m fi di)
(ttable o si m fi di)
(ttable o s o)
(ttable fi s o)
(ttable o oi o fi di si = s d f oi)
(ttable fi m m)
(ttable fi o o)
(ttable fi di di)
(ttable fi si di)
(ttable di fi di)
(ttable di di di)
(ttable di si di)
(ttable si fi di)
(ttable si di di)
(ttable fi f f = fi)
(ttable di p p m o fi di)
(ttable si p p m o fi di)
(ttable si s si = s)
(ttable s p p)
(ttable s m p)
(ttable d p p)
(ttable d m p)
(ttable f p p)
(ttable s o p m o)
(ttable s fi p m o)
(ttable s di p m o fi di)
(ttable s si si = s)
(ttable d o p m o s d)
(ttable d fi p m o s d)
(ttable f m m)
(ttable f o o s d)
(ttable f fi f = fi)
(ttable oi p p m o fi di)
(ttable mi p p m o fi di)
(ttable oi m o fi di)
(ttable oi o o fi di si = s d f oi)
(ttable mi m si = s)
)
```

(This ordering tries to illustrate the natural grouping of entries in the table. Some entries are omitted: in particular, those involving equality follow by ordinary logical principles; some entries provide no information; and the entire table is symmetric so only half of it is necessary.) More discussion of the topological structure of the table can be found in (Freska 1992), who develops a very intuitive iconic notation.

The table can be thought of as defining an algebra on subsets of the set AR = {`p,m,o,s,d,f,fi,di,si,oi,mi,pi,=`} of thirteen relation *names*,. Temporal reasoning can then be performed by computing multiplication in this algebra, where an empty string indicates a contradiction. This has been a very influential technique for interval reasoning. Vilain and Kautz (1986) have shown that computing closure in this algebra is an NP-complete problem in general, although useful polynomial time algorithms have been developed for special cases (Vilain, Kautz & Van Beek 1989)

This algebraic approach to interval reasoning is rather different in spirit than the simple "reduction to endpoint" definitions given below. For example, consider the relationship between two intervals `?i` and `?j` which is that the first begins before the second ends, ie `(before (beginof ?i)(endof ?j))`. To express this in the algebraic style one must consider which of the thirteen relationships is consistent with this arrangement and list them all explicitly. The answer is `{p, m, o, s, d, =, fi, si, fi, di, oi}`, ie all eleven of these are consistent with that constraint on the endpoints. This awkwardness of the disjunctive-list notation is due to the fact that each particular interval relation commits itself to the relative positions of both endpoints, so a lack of commitment can only be expressed by a disjunction.

Several entries recur in the table several times: the sequences `(p m o fi di)` and `(o s d f = fi di si oi)` for example. Freska notes that several of these have natural interpretations as weaker constraints on interval endpoints. For example, `(p m o fi di)` is the relation of beginning earlier than, with no constraint on the relations between the ends of the intervals; while the second disjunction is the relation `ncdf` in section 4.1. The thirteen relations can be defined from these by conjunction rather than disjunction. (For example, if we also define *exact* to mean that the two intervals' endpoints are somewhere exactly aligned (this is `(m s f = si fi mi)`), then meets is *beginsearlierthan* and *not ncdf* and *exact*.) This suggests the possibility of finding such a set of weaker relations which is also closed under transitivity, which would greatly simplify the task of computing transitive closure.

One might have hoped for a simpler summary of the relations between such an intuitively acceptable set of temporal relationships. They can be appropriately defined in simpler theories. A theory needs to provide enough axioms to conclude that these relations are an exhaustive and exclusive set of interval relations, and to derive the transitivity table. We give two ways of doing this.

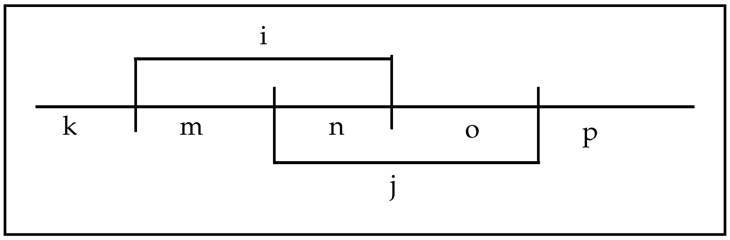
4.4 Thirteen relations in terms of meets

These definitions use one basic formal trick, which is to force two intervals to have a common endpoint by hypothesising that a third interval exists which meets them both, or which they both meet. This is rather like the familiar juggling feat of holding boxes in the air by clamping them between two boxes held in the hands.

The intuitive content of these definitions is probably best shown by drawing the linear patterns of intervals named in the quantifiers. `Overlaps` is the most complicated case.

```
;;precedes-IM
(defrelation precedes (?i ?j) :=
              (exists (?k) (meets ?i ?k ?j)))

;;overlaps-IM
(defrelation overlaps (?i ?j) :=
                   (exists (?k ?m ?n ?o ?p)
                      (and (meets ?k ?m ?n ?o ?p)
                           (meets ?m ?j ?p)
                           (meets ?k ?i ?o) )))
```




```
;;starts-IM
(defrelation starts (?i ?j) :=
             (exists (?k ?m ?n)
                     (and (meets ?k ?i ?m ?n)
                          (meets ?k ?j ?n) )))

;;during-IM
(defrelation during (?i ?j) :=
             (exists (?k ?m ?n ?o)
                     (and (meets ?k ?m ?i ?n ?o)
                          (meets ?k ?j ?o) )))



;;finishes-IM
(defrelation finishes ( ?i ?j) :=
             (exists (?k ?m ?n)
                     (and (meets ?k ?m ?i ?n)
                          (meets ?k ?j ?n) )))
```

Allen & Hayes (1987) show how the entire 13 x 13 transitivity table of these relations and their inverses can be derived within the theory of 'meeting' described earlier plus these definitions.

5. Points and Intervals

In this section we consider putting together the ideas of point and interval into combined theories.

5.1 Thirteen relations in terms of endpoints

The thirteen relations can be defined directly, following their intuitive meanings, by using the point ordering relation `before` applied to the endpoints of the intervals. To do this we introduce two functions `beginof` and `endof` from intervals to points. The inverse function `between` is also useful. These axioms and definitions, added to the theory LINEAR-POINT, constitute a theory we will call ENDPOINTS:

```
;;beginof-endof-EP
(forall (?i)(=> (timeinterval ?i)
                        (and (timepoint (beginof ?i) (endof ?i))
                             (before (beginof ?i) (endof ?i))
                        )
        ))

;;between-EP
(forall (?p ?q)(<=> (before ?p ?q)
                    (and (= ?p (beginof (between ?p ?q)))
                         (= ?q (endof (between ?p ?q)))))))

;;precedes-EP
(defrelation precedes (?i ?j):=
             (and (timeinterval ?i ?j)
                  (before (endof ?i) (beginof ?j))))

;;overlaps-EP
(defrelation overlaps (?i ?j):=
             (and (timeinterval ?i ?j)
                  (before (beginof ?i) (beginof ?j) (endof i))))

;;starts-EP
(defrelation starts (?i ?j):=
         (and (timeinterval ?i ?j)
              (= (beginof ?i) (beginof ?j))
              (before (endof ?i) (endof ?j)) ))
;;
;;during-EP
(defrelation during (?i ?j):=
          (and (timeinterval ?i ?j)
               (before (beginof ?j)
                       (beginof ?i)
                       (endof ?i)
                       (endof ?j)) ))

;;finishes-EP
(defrelation finishes (?i ?j):=
         (and (timeinterval ?i ?j)
              (before (beginof ?j) (beginof ?i))
              (= (endof ?i) (endof ?j)) ))
```

The entries in the transitivity table can be straightforwardly derived within ENDPOINTS. However, we can also take these axioms, add them to the theory INTERVAL-MEETING, and regard `between-EP` as a definition of `before`; and then all the axioms of LINEAR-POINT are derivable. So this theory provides a kind of bridge between the point-ordering view of time and the interval-meeting view, from the glass continuum perspective.

Notice that `beginof-endof-EP` means that there cannot be a single-point interval in this theory. If we allowed single-point intervals then the transitivity table would no longer be derivable. For example, it would then be possible to have three intervals such that I meets J meets K and I meets K. Moreover, one interval could both meet and start another. Later we will consider a variation in which this restriction is removed, and which reconciles these apparently unintuitive consequences.

Models

All models of LP , including the nonstandard ones, extend immediately to models of EP by interpreting intervals in any of the three ways mentioned in section 4.2 above. These axioms fit most naturally into the glass continuum, since they make no distinction between open and closed timeintervals; but as noted there, it is possible to interpret timeintervals as half-closed intervals in the point continuum.

Some care is necessary in considering models over the rationals. We cannot take the universe to be *all* rational intervals, since there are rational intervals with no rational endpoints. The universe of models of EP must be made up from intervals with rational endpoints. Perhaps not surprisingly, therefore, when we insist that intervals are attached to their ends, we can only have countably many of them.

5.2 Open and closed intervals: the point continuum

Although the thirteen relations can be interpreted in the point continuum, difficulties arise if we combine the mathematical intuitions from real analysis with these axioms. For example, Galton (1990) argues that Allen's axiom for 'property negation', when applied to a smoothly moving body, seem to imply that it is never at any position (since it is never at any position for a whole subinterval, `(pnot (at ?x))` is always true throughout any interval). This argument however assumes that "interval"means mathematical interval, and indeed Galton goes on to develop a variation on Allen's theory which is as directly interpretable in the point continuum as Allen's is in the glass continuum. We give a variation here which is equally expressive but somewhat less complex.

Galton distinguishes two kinds of proposition: a *state of position* and a *state of motion*. Paradigmatic examples are respectively, a ball being at rest and a ball being in motion. Galton however is careful to distinguish being at rest from having zero velocity. For example, a ball tossed in the air has zero velocity at the single timepoint when it reaches the top of its parabola, but it is not *at rest* there; this is a state of motion but not of position.

Galton gives many axioms relating states of position and motion, using a complicated variation of the 'holds' notation discussed in section 2 earlier. However, since states of position are those things that can hold true during a closed interval, while states of motion must hold during open intervals, the two classes of proposition must be exclusive, and to distinguish them it is sufficient to distinguish the kinds of interval during which they hold. If we further think of an interval as consisting of the points it contains, we can easily define an equivalent language which has as primitive just the notion of `holds at` a point. As described earlier, with this simple interpretation 'holds' can be transparently removed from the language, and such axioms as

```
**     (<=> (holds-at (not ?p) ?i) (not (holds-at ?p ?i)) )
```

become tautologous.

The distinction between states of motion and position can then be reinterpreted as differences between different kinds of relation. Some can only be relativised to open intervals, others only to closed. For example, the property of *moving* is taken to be a relation between an object and a closed interval, while *resting* becomes a relation to an open interval. When something starts to move, the two intervals of its resting and moving `meet`, but the latter contains the meeting point, where the motion has begun but the velocity is zero. For the old example of the light, either one or the other of *being lit* and *not being lit* has to be a state of motion; or else, perhaps more plausibly, they can both be states of rest, but then the single meetingpoint is where a special state of motion predicate – <u>goingout</u> or <u>comingon</u> , or perhaps simply <u>changingstate</u> – is true.

There are several ways to obtain a theory for open and closed intervals. One method is to define intervals as sets of points, and use set-theoretic comprehension principles to establish that appropriate intervals exist. We do not develop this standard mathematical approach here, but give selfcontained axiom systems which can be connected to set theory later, if required. There are in any case some complications in thinking of intervals literally as sets of points.

The most straightforward way is probably to talk of endpoints, as in the theory EP. I will use the relation `in` between a point and a containing interval (this is not the same as `during`) and `open` and `closed` as predicative relations on intervals.

```
;;PC-in-syntax
(forall (?x ?y)(=> (in ?x ?y)
                   (and (timepoint ?x)
                        (timeinterval ?y)) ))

;;open-close-in-PC
(forall (?i)(=> (timeinterval ?i)
        (or (and (open ?i)
                 (not (closed ?i))
                 (forall (?p)(<=> (in ?p ?i)
                                  (before (beginof ?i) ?p (endof ?i))
                 ))  )
             (and (closed ?i)
                  (not (open ?i))
                  (forall (?p)(<=> (in ?p ?i)
                                   (bbefore (beginof ?i) ?p (endof ?i))
                  ))  ))))
```

So far nothing establishes that intervals actually exist, but we can assert this directly.

```
;;begin-end-PC-1
```

```
(forall (?p ?q)(=> (timepoint ?p ?q)
        (or  (before ?q ?p)
             (exists (?i)
                     (and (timeinterval ?i)
                          (closed ?i)
                          (= (beginof ?i) ?p)
                          (= (endof ?i) ?q)
                     )
              ))
        )
)
```

The rather awkward disjunction here is to allow the case where ?p and ?q are the same point. The similar axiom for open intervals reads more naturally:

```
;;begin-end-PC-2
(forall (?p ?q)(=> (timepoint ?p ?q)
        (<=>  (before ?p ?q)
              (exists (?i)
                      (and (timeinterval ?i)
                           (open ?i)
                           (= (beginof ?i) ?p)
                           (= (endof ?i) ?q)
                       )
              ))
        )
)
```

The skolem form of this axiom provides the useful function `between` from a pair of points to the open interval between them.

```
;;between-PC
(defunction between
(forall (?p ?q)(=> (timepoint ?p ?q)
        (<=> (bbefore ?p ?q)
             (and (timeinterval (between ?p ?q))
                  (= ?p (beginof (between ?p ?q))
                  (= ?q (endof (between ?p ?q))  ))))))
)
```

The biconditional here makes the very strong assumption that every pair of points defines an interval and vice versa. Weaker "bindings" between interval and points could also be considered, for example by replacing this by a simple implication, or restricting the quantifier to a special subset of 'endpoints'. For example, one might want to consider a dense theory of points but only allow a discrete universe of intervals, so that some points had no interval between them. I will not explore such ideas in detail, however.

Every open interval has a closed interval with the same endpoints. The skolem form of this statement provides the function `closure`. This axiom would be trivial in the glass continuum, but needs to be explicitly stated here:

```
;;closure-PC
(deffunction closure
(forall (?i)(=> (timeinterval ?i)
        (and  (timeinterval (closure ?i))
```

```
              (closed (closure ?i))
              (= (beginof ?i) (beginof (closure ?i)))
              (= (endof ?i) (endof (closure ?i)))
        )))

)
```

It follows that an interval is closed just when it is equal to its own closure, as expected.

A moment is an interval which is as short as possible, ie has no points inside it :

```
;;moment-PC
(defrelation moment (?i) :=
       (forall (?p)(<=> (in ?p ?i)(or (= ?p (beginof ?i))
                                      (= ?p (endof ?i)) ))))
```

The definitions of the thirteen relations used in 4.5 above now need to be reconsidered. (As they were inspired by the glass continuum, this is perhaps not surprising.) Only a closed interval can meet an open one, and vice versa; and a subinterval can start or finish only an interval of the same kind. The concept of being the same kind as, ie <u>also closed or also open</u>, or `acoao`, will be useful here:

```
;;acoao-PC
(defrelation acoao (?i ?j)(=> (timeinterval ?i ?j)
             (or  (and (open ?i) (open ?j))
                  (and (closed ?i) (closed ?j))
                  )))
```

Then the three cases where endpoints are supposed to be precisely aligned can be rewritten thus:

```
;;meets-PC
(defrelation meets (?i ?j) :=
          (=> (timeinterval ?i ?j)
              (and (not (acoao ?i ?j))
                  (= (endof ?i) (beginof ?j)) )))

;;starts-PC
(defrelation starts (?i ?j) :=
         (=> (timeinterval ?i ?j)
             (and  (acoao ?i ?j)
                   (= (beginof ?i) (beginof ?j))
                   (before (endof ?i) (endof ?j))
                   )))

;;finishes-PC
(defrelation finishes (?i ?j) :=
        (=> (timeinterval ?i ?j)
            (and  (acoao ?i ?j)
                  (before (beginof ?j) (beginof ?i))
                  (= (endof ?i) (endof ?j)) )))
```

The other Allen relations are defined just as in ENDPOINT.

If these axioms are added to the basic point-order theory LINEAR-POINT, the resulting theory is a sketch of the usual mathematical account of the rational line or the real continuum.

It does not have continuity assumptions, however, since these are essentially not first-order. Whether they are needed is an interesting, and ultimately empirircal question. If this theory is being used in a scientific or engineering domain then certain *consequences* of continuity are certainly useful, and may be essential: notably, for example, the ijntermediate-value theorem and the existence of solutions to certain classes of equations. However, these can often be stated directly. In any case, I do not here consider this issue (which goes beyond the narrow question of temporal effectiveness) in detail.

Two open intervals cannot meet, so if two open intervals `(between a b)` and `(between b c)` share an endpoint, then there is a single-point closed interval just separating them, so that `(between a b) meets [b,b] meets (between b c)`. These single-point intervals are the moments.

Moments are closed, and every closed single-point interval is a moment, as we would expect. In stark contrast to the axioms in 5.1, these single-point moments are everywhere: by `between-end-PC-1`, <u>every</u> point occupies one. The biconditional in `between-end-PC-2` ensures that a single-point open interval is impossible.

The chief utility of moments until now has been to state a discreteness condition. But because a moment now can consist of nothing but a point, axioms which assert that intervals have adjacent moments will be true even in the real line, thus failing completely to guarantee an underlying discrete structure of seperate time-ticks. We cannot insist that a moment `meet` another moment since this is provably impossible; being both closed, their meeting is excluded by the acoao condition. The correct way to state density or discreteness is in terms of points rather than intervals, since the intervals in this theory are defined by the locations of their endpoints. We can simply add the appropriate definitions in the extended theories DENSE- or DISCRETE- LINEAR-POINT.

<u>Models</u>

> These axioms have a curious consequence in a discrete model of time, where there are points `p,q` ordered by `before` but with *no* points between them; therefore there are open timeintervals `(between p q)` which contain no timepoints at all. The discrete timeline consists of these empty open intervals interleaved with single-point moments containing the timepoints. This rather strange (although consistent) picture arises from combining two rather different intuitions: the universal quantifier in 5.2.6 is suggested by a vision of smooth continuity, which the discreteness axiom explicitly denies.
>
> This would be a contradiction in a set-theoretic model, where these would all be the empty set and hence be identical, forcing time to be circular in a particularly pathological fashion. However, a set-theoretic development would not need these axioms; and we do not need to identify these empty moments, since they are distinguishable by their endpoints (which, being open, they do not contain). For example, one discrete model of these axioms is provided by the integers where `timepoint` is true of the even integers, a `timeinterval` is a connected sequence of integers which is `open` when its ends are odd and `closed` when they are even, and `meets` is simply adjacency. The ‘empty’ open intervals are then the odd integers.
>
> The density axiom produces more intuitive results, and this extension of the theory has the usual models. With the density assumption, a moment must contain only a single point which is both its end and its beginning.

Galton’s distinction between states of motion and rest is now simply the distinction between the kind of interval they can be said to hold during. They both `hold-in` an interval just when they `hold at` all the points `in` the interval; but if the interval is closed, then they are a state of motion, and if open, a state of rest. Things that `hold-in` single-point moments (as opposed to holding at a point) must therefore be states of motion.

5.3 The Vector Continuum: placing points in glass

The different intuitions about the continuum give rise to very different axioms. However, it is possible to combine some of the most useful features of both of them into a single consistent framework.

First, we simply add the notion of a point to the description of the glass continuum given in 4.2, ie the theory INTERVAL-MEETING. To preserve the intuition, however, the basic relation between points and intervals is not that of containment – the `in` relation of 5.1 – but that of a *meeting-point*: points are the places where intervals meet. We will express this by a three-way relation `meets-at` between two meeting intervals and a point.

The most natural relation between interval and point is expressed here by the relation meets-at which is true when a point is the meeting-place of two intervals:

```
;;VC-syntax-meets-at
(forall (?x ?y ?z)(=>(meets-at ?x ?y ?z)
                     (and (timeinterval ?x ?z) (timepoint ?y))))

;;meets-at-VC
(defrelation meets-at (?i ?p ?j) :=
               (= ?p (endof ?i) (beginof ?j)))
```

It follows therefore that `(meets ?i ?j)` if and only if `(exists (?p)(meets ?i ?p ?j))`.

If we now define the timepoint ordering :

```
 (defrelation before (?p ?q) :=
          (exists (?i ?j ?k)
               (and (meets-at ?i ?p ?j)(meets-at ?j ?q ?k)))  )
```

then its properties follow from the axioms of INTERVAL-MEETING: that is, LP can be derived within IM+these three axioms. (Notice again the importance of the infinity assumption , which is now necessary to establish that every interval has endpoints.) This theory, like ENDPOINTS, therefore provides a bridge between the point and interval ways of conceptualising the glass continuum. To emphasise the point; if we define `beginof` and `endof` correctly, the definitions of the thirteen relations in the style of the interval theory IM, or that of the endpoint theory EP become provably equivalent:

```
(defrelation beginof (?i) := (exists (?j)
                   (meets-at ?j (beginof ?i) ?i)))

(defrelation endof (?i) := (exists (?j)
                   (meets-at ?i (endof ?i) ?j)))
```

The resulting theory is still openminded about density or discreteness, and can be pushed in either direction by adding suitable extension axioms, eg by extending LP or IM.

A more interesting alternative, however, allows the idea of intervals with a direction. These will be useful when talking of durations, since it is natural there to think of a negative duration as a 'debt' of time; and it results in a very elegant simplification of many of the axioms, overcoming some artificial restrictions which have been necessary in order to make the axioms have the needed conclusions.

Consider the axiom `beginof-endof-EP`:

```
(forall (?i)(=> (timeinterval ?i)
                     (and (timepoint (beginof ?i) (endof ?i))
                          (before (beginof ?i) (endof ?i))
                     )
        ))
```

and consider the effect of weakening it to remove the restriction that the beginning of an interval is `before` its end.:

```
(forall (?i)(=> (timeinterval ?i)
                (timepoint (beginof ?i) (endof ?i))
        ))
```

This allows intervals with are 'pointed backwards' as well as interval with idenical beginning and endings, ie single-point intervals. The lack of single-point intervals is a distinct weakness of the 'glass continuum' theories such as ENDPOINTS, compared to the POINT-CONTINUUM, but this suggests an alternative way to incorporate them.

```
;;moment-VC
(forall (?x) (<=> (moment ?x)
                  (= (beginof ?x)(endof ?x))))
```

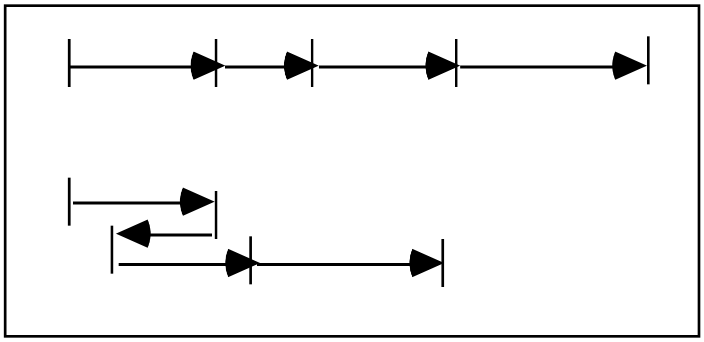
Here then, an interval is defined by *any* two points. If the beginning is `before` the end, then we will say that the interval is `forward`, and given any interval we will have a function `back` which 'reflects' it.

```
;;forwards-VC
(defrelation forwards (?i) := (before (beginof ?i)(endof ?i)))


;;back-VC
(deffunction back (?i) = (between (endof ?i)(beginof ?i)))
```

That `(back (back ?x)) = ?x` then follows from the definition of `between` in `ENDPOINTS`.

Actually this 'mirror' analogy is somewhat misleading, since a backward interval should not be thought of as *going* backwards in time - time itself does not move, of course - but rather as a debt of time or an amount owing. For example, a sequence of meeting intervals represents a longer interval which is their sum; to include a backward interval in such a sequence would simply be to *diminish* the total time it represented:



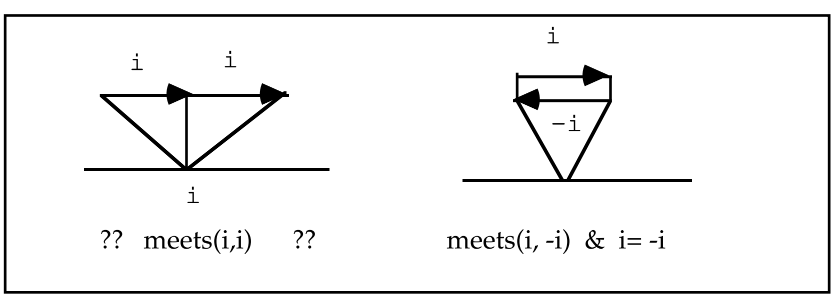
This is rather like observing that some event is earlier than expected.

Similarly, a single-point interval - a moment - added to another interval should not affect its duration at all. There seems now, in fact, to be little distinguish a moment from a timepoint, other than our insistence that the two categories are disjoint. The motivation for such an insistence comes from the idea that intervals, but not points, are the things during which propositions are true, or when events happen. This intuition was fundamental to Allen's development and has motivated others; and it was precisely this (considered more as a bug than a feature) which led Galton to explicitly deny the glass continuum intuition and return to the point continuum. There seems to be little more than apoint when the ball is motionless at the top of its parabola, for example.

This all suggests that we take the unusual step of allowing a moment to be both an interval and a point. The categories are disjoint everwhere else, but here they overlap:

```
;;VC-moment
(forall (?x) (<=> (moment ?x)
                  (and (timepoint ?x)
                     (timeinterval ?x))))
```

Moments now have some interesting properties. Following the axioms in `INTERVAL-MEETING` and the definitions in `LINEAR-POINT`, it is easy to show that a moment meets itself: indeed, that it meets itself at itself. This seems extremely unintuitive until we observe that any interval meets its own back-reflection. We can think of a moment not as the limiting case of two copies of itself placed one before the other - which is clearly impossible - but rather as the limiting case of an interval meeting its own reflection:



It now also follows that if two (nonmomentary) intervals meet at a moment, then they also meet. This means that a moment can be placed 'between' two meeting intervals without blocking their meeting, in marked contrast to the situation in the point continuum.

In many ways this seems more natural. It allows us to describe a pattern of meeting intervals without being overly concerned about whether or not the meeting-places are worthy of mention. Deciding that a meetingplace is a moment (and hence that something can be true there but nowhere else, as in the tossed ball) would not require massive updating of a set of assertions, rewriting all the 'meets' to be 'precedes', re-evaluationg the open/closed status of intervals, and so forth, but can be transparently added, and is inconsistent only with an explicit denial of its status as momentary..

This provides the most satisfying framework for describing the glass continuum. A moment now is a both a timepoint and a timeinterval. Since it meets other intervals, it can also take part in the other interval relationships defined in INTERVAL-MEETING. It is (when considered as a point) both the `beginof` and `endof` itself (when considered as an interval). In some ways therefore it acts similarly to the one-point intervals in the point continuum. For example, we could distinguish the interval `between` two such point-moments from the result of adding them to its ends, and regard this addition as something like the operation of closure in the point continuum. But many intervals need not end in such moments, so 'closure' might be rarely possible: and in any case, these 'closed' intervals do not behave any differently than their 'interiors' in how they relate to their neighbors. Points here are still not as substantial as they are in the point continuum, even when given the status of being intervals.

This also fits quite well with the intuition of an interval as giving information about the exact time of a timepoint. Since a moment is a point, it has no other points within it, hence it is an interval which identifies a point as precisely as possible. It represents the limits of our abilities to measure exactly when something happens. In the dense case, this would seem naturally to be the point itself.

5.3.1 Approximate-meeting

Another way to motivate this way of describing time comes from the following essentially model-theoretic argument. Return for the present to the older idea of moment as a shortest possible interval, but *not* a point, as described in the theory INTERVAL-MEETING. Let us say that two intervals *approximately meet* if they are separated only by a moment:

```
;;A
(defrelation ameets (?i ?j):=
                    (exists (?k)
                          (and (moment ?k)(meets ?i ?k ?j)) ) )
```

and then allow this as one way of meeting:

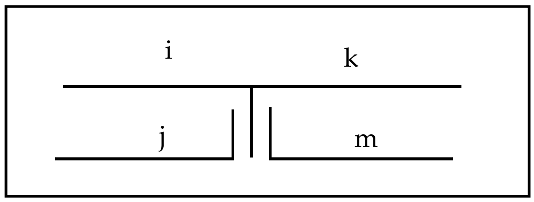
```
;;B
(defrelation mmeets (?i ?j):=
                    (or (ameets ?i ?j)(meets ?i ?j)) )
```

(The name should suggest a momentary hesitation) Now, if we assume that two moments cannot meet:

```
;;C
(forall (?i ?j) (=>
             (and (moment ?i)(moment ?j))
                  (not (meets ?i ?j)) ))
)
```

then `mmeets` satisfies the axioms of INTERVAL-MEETING (IM) . That is, if we rewrite the IM axioms with `mmeets` in place of `meets`, the resulting theory – call it AIM – is deducible within IM + A,B,C. (Hayes & Allen 1991)

(The reason for the no-meeting requirement can be seen by considering the 'place' axiom with `ameets` instead of `meets`. The assumptions allow this to occur:



But now the intervals j and m do not even approximately meet, since there are two moments between them. If tiny errors are allowed to add up, they become large enough to notice.)

This means that `meets` could consistently have been interpreted this way all along (assuming still that moments cannot meet). Any model of IM + A,B,C is automatically a model for AIM; and any model of AIM is also one of IM where `meets` is sometimes interpreted as `ameets`. Therefore we can <u>merge</u> the relations `meets` and `ameets` into a single relation without confusion, retaining all truths. If we call this combined relation `meets`, then this amounts to saying that some meetings can be at moments without violating the theory. The point-moment models described in section 4.2, in which a subset of points in the model were singled out as standing for moments, were indeed such interpretations, with the 'isolation' of the moment-points corresponding exactly to the no-meeting axiom C. The effect of this merging can be described as identifying moments with points.

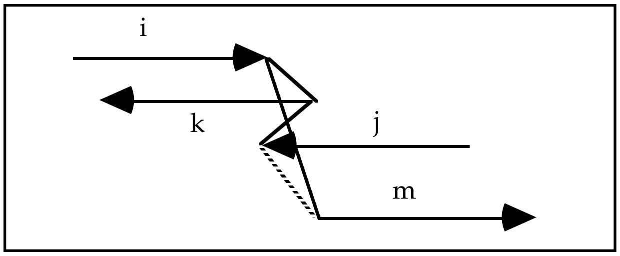
It would be nice if we could simply conjoin all the axioms in INTERVAL-MEETING and ENDPOINTS, but it is now necessary to adapt some of the earlier axioms slightly, since point-moments will no longer serve to establish a space between meetingpoints:

```
;;beginof-endof-VC (generalises EP)
(forall (?i)(=> (timeinterval ?i)
                (and (timepoint (beginof ?i) (endof ?i)))))

;;between-VC (generalises EP)
(forall (?p ?q) (and (= ?p (beginof (between ?p ?q)))
                     (= ?q (endof (between ?p ?q))))))

;;plus-VC
(deffunction plus
(forall (?i ?j)(=>(meets ?i ?j)
                  (= (plus ?i ?j)
                     (between (beginof ?i)(endof ?j)))))
)
```

The axioms of INTERVAL-MEETING adapt quite well to the generalisation which allows backwards and pointlike intervals. For example, the ‘place’ axiom is still true even when some of the intervals mentioned are backwards:



The definitions of the Allen relations given in ENDPOINTS work here perfectly well. The definitions in INTERVAL-MEETING, however, need to be restricted to forward intervals, or they fail to make the relevant distinctions. Here, the mere existence of an interval is not sufficient to ensure the past-to-future ordering of its endpoints.

6. Durations

A duration is a property of an interval, or equivalently of a pair of points defining the ends of the interval. Since the duration of a point (including a point-moment) is zero, we can here be somewhat more careless about the distinctions between different views of the continuum, and between open and closed intervals. Any theory of duration ought to apply to both views equally well; and, except where noted, these axioms can be added to any of the earlier theories. The conclusions can often only be reached in one of the more comprehensive theories, however. The distinction between dense and discrete time is often important since in dense time, a moment has no duration, but in discrete time it must have some.

6.1 Basic properties of durations

It would be acceptable to assume immediately that durations were, say, real numbers; but in the spirit of the earlier sections I will develop the theory with the minimal assumptions necessary. Whatever durations are, some things seem clear. Durations can be compared; durations can be added together (since the duration of two meeting intervals is the sum of their durations); there is a zero duration (which is the duration of a point); and finally, clocks measure duration by counting , so it must be possible to multiply durations by integers. These three basic assumptions are embodied in the constant `zero`, the chained transitive relation `less`, and the functions `add` and `times` used in these axioms. The type predicate is `timeduration`, and `duration` is a function from intervals to timedurations.

```
;;DU-syntax
(and (predicative timeduration)
     (forall (?x ?y)(and
                (=> (exist (?z) (= ?z (mult ?x ?y)))
                    (and (integer ?x) (timeduration ?y)))
                (=> (exist (?z) (= ?z (add ?x ?y)))
                    (timeduration ?x ?y))
                (timeduration zero)  ))


;;mult-DU
(deffunction mult
(forall (?d ?n ?m)
        (and
         (= (mult 0 ?d) zero)
         (= (mult 1 ?d) ?d)
         (= (mult (+ ?n ?m) ?d))
            (add (mult ?n ?d) (mult ?m ?d)))
        ))
)

;;add-DU
(deffunction add
(forall (?d ?e @f)
        (and (= (add zero ?d) ?d)
             (= (add ?d ?e) (add ?e ?d))
             (= (add ?d (add ?e @f)) (add (add ?d ?e) @f))))
)
```

Several different functions satisfy this axiom. Since in general there is no notion of a unit duration, addition cannot be defined recursively in terms of a successor function.

```
;;less-DU
(forall (?d ?e)(<=> (positive ?d)(less ?e (add ?d ?e))))
```

The relation positive cannot be defined purely in the duration theory, but requires the notion of a forward interval, which in turn is defined in terms of the basic before relation on timepoints. Without reference to before, times,and hence timedurations, are completely symmetric with regard to the direction of time.

```
;;numberin-DU
(deffunction numberin
(forall  (?d ?e)
         (=>
          (exist (?n)(and (integer ?n) (= (mult ?n ?d) ?e)))
          (= (mult (numberin ?d ?e) ?d) ?e)))
)
```

`Numberin` is a useful function when we can be sure that copies of one duration fit exactly the another.

6.2 Intervals, points and durations

The most basic facts about the duration of intervals is that they add up properly:

```
;;duration-DU-plusadd
(forall (?i ?j)
        (=> (meets ?i ?j)
            (= (duration (plus ?i ?j))
               (add (duration ?i) (duration ?j)) ))))
```

and that points have no duration:

```
;;duration-DU-zeroduration
(forall (?x)(<=> (timepoint ?x) (= (duration ?x) zero)))
```

Examples of durations include

```
;;DU-examples
(and  (timeduration year)
      (timeduration week)
      (timeduration day)
      (timeduration hour)
      (timeduration minute)
      (timeduration second) )
```

We might also want to insist that nontrivial moments cannot have zero duration. In VC, since moments are timepoints, this covers them. If not, however, the following might seem a reasonable way to do it:

```
???   (forall (?i:timeinterval)(=> (= (duration ?x) zero)
                                    (moment ?i) ))
```

but in simple discrete theories it is false, while in simple dense theories it says nothing , since moments do not exist. A suitable axiom for general use talks about points:

```
(forall (?p ?q)(<=>
             (before ?p ?q)
             (not (= (duration (between ?p ?q)) zero)) ))
```

The biconditional guarantees that moments in ENDPOINTS have zero duration, for example, and that one-point closed intervals in the point-continuum theory also do.

(One might wish to allow non-pointlike intervals with zero duration, by weakening the biconditional in `duration-DU-zeroduration` to a simple conditional. These things would complicate the description of clocks, however, so here we assume that they are impossible.)

If time is totally ordered and we select a certain fixed starting timepoint, then any other point is uniquely defined by the duration of the interval between it and the start time. So timepoints can be <u>identified</u> by durations:

```
;;duration-DU-rigid
(forall (?i ?j)
        (=> (and (timeinterval ?i ?j)
                 (= (beginof ?i) (beginof ?j))
                 (= (duration ?i) (duration ?j)) )
            (= (endof ?i) (endof ?j)) ))
```

Notice that this does not claim that the interval itself is *uniquely* identified, since open and closed intervals have the same duration in the point continuum. However, in theories where endpoints identify intervals this will of course suffice. Also, rigidity does not hold in branching time models, where the clock or calendar time of a timepoint fails to uniquely specify it since there are many alternative timelines all going at the same rate, as it were. A convenient variation on the rigidity axiom uses the function `from`, from a point and a duration to the point that much later:

```
;;from-DU
(deffunction from
(forall (?p ?d) (= ?d (duration (between ?p (from ?d ?p)))))
(forall (?p ?d)(=> (and (timepoint ?p)(timeduration ?d))
                   (timepoint (from ?p ?d))))
)
```

Backwards and reflected intervals have negative and negated durations:

```
;;positive-DU
(defrelation positive
(forall (?p ?q)
        (<=> (before ?p ?q)
             (positive (duration (between ?p ?q))) ))

;;DU-back
(forall (?i)(= zero (add (duration ?i)(duration (back ?i)))))
```

Some other useful functions include a function which totals a sequence of durations

```
;;total-DU
(deffunction total
(forall (?d @s)
        (and  (= (total ?d) ?d)
              (= (total ?d @s)
                 (add ?d (total @s)) )))
```

```
)
```

and one that forms a sequence of the durations of a sequence of intervals, a kind of duration-on-sequences function:

```
;;dduration-DU
(deffunction  dduration
(forall (?x @l)(=> (timeinterval ?x)
                   (and (= duration ?x)(dduration ?x))
                        (= (dduration ?x @l)
                        (listof (duration ?x) (dduration @l)))))))
```

6.3 Simple Clocks

A simple clock is characterised by a `start`, which is a timepoint, and a `beat`, which is a duration. It works by counting the number of beats between its start time and the time being measured. A clock makes a continuous timeline seem discrete by mapping every time to a `clocktick`.

```
;;simpleclock-CL
(forall (?c) (=>(simpleclock ?c) (and (duration (beat ?c))
                                 (timepoint (starttime ?c)))))
```

```
;;clocktick-CL
(defrelation clocktick (?p ?c) :=
            (exists (?n)(and (integer ?n)
                             (= (duration (between (starttime ?c) ?p))
                                (mult ?n (beat ?c)) ))))
```

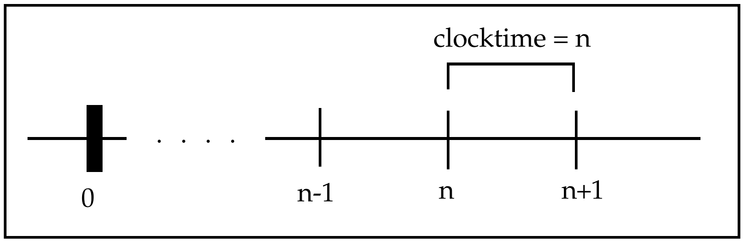
hence

```
 (forall (?n ?c)
      (clocktick (from (times ?n (beat ?c)) (starttime ?c)) ?c))
```

Since intervals can be backwards, a clock 'tells' the time even before its starttime, but such clocktimes are negative.

A (simple) clock time is the time of a point as measured by the clock, which is equal to the time of the immediately preceding clocktick:

```
;;simpleclocktime-CL
(defunction simpleclocktime (?p ?c)
  (and  (integer (simpleclocktime ?p ?c))
        (less (mult (simpleclocktime ?p ?c) (beat ?c))
              (duration (between (starttime ?c) ?p)))
        (not (less (mult (+ 1 (simpleclocktime ?p ?c)) (beat ?c))
                   (duration (between (starttime ?c) ?p))  ))
        )
  )
```



This is the best we can do in general, since no clock can precisely locate every point in a dense timeline. In the discrete timelines, however, we might hope for more accuracy.

Until now nothing has insisted that every moment of a discrete time must have the same duration, and indeed this would not be an appropriate claim to make in all theories (Situation-Calculus Time for example). However, now seems the right time to make this insistence. If we call this universal atomic amount of time `quantum`, then we have simply:

```
(forall (?i) (<=> (moment ?i)
             (= (duration ?i) quantum) ))
```

(In some continuous time theories, this means that `quantum` is `zero`; in others, it says nothing since there are no moments.)

We can reasonably expect that a clock-beat is some definite number of quanta, since when time is discrete there are no other times available for the clock to tick at. Indeed, if time is discrete then every duration is made of quanta:

```
(forall (?d)(<=> (timeduration ?d)
                (exists (?n)
                        (= ?d (mult ?n quantum)) ) ))
)
```

and so it follows that a clock which could beat at the pulse-rate of the universe could indeed serve as a universal clock:

```
;;quantumclock-Q
(defrelation quantumclock (?c) :=
             (and (simpleclock ?c)(= (beat ?c) quantum))))

;;simpleclocktime-Q
(forall (?c ?p)
        (=>  (and (quantumclock ?c)(timepoint ?p))
             (= (mult (simpleclocktime ?p ?c) (beat ?c))
                (duration (between (starttime ?c) ?p))))))
```

Clocks are shift-invariant. This is easier to say in KIF than in English:

```
;;CL-shift-invariance
(forall (?c ?d)
       (=> (and (= (beat ?c) (beat?d))
                (clocktick (starttime ?d) ?c))
```

```
        (forall (?p)(=>(timepoint ?p)
                       (= (simpleclocktime ?p ?c)
                          (+ (simpleclocktime ?p ?d)
                             (simpleclocktime (starttime ?d) ?c)))))
    ))
```

Models

Again, the semantic weight of these axioms is closely tied to the possible interpretations of the arithmetic terms that occur in them. If `integer` really refers only to integers, then all the nonstandard models of the timeline are ruled out and these axioms, when added to any of the earlier discrete-point theories, have only standard interpretations. However, nonstandard models of arithmetic adapt perfectly well to give nonstandard clocks, which might be called pink-rabbit clocks, since they beat forever and then keep on going.

6.4 Calendars

A calendar is a fixed system of timeintervals and subintervals which divide the timeplenum into separate, identifiable pieces. A clock defines a calendar, but not all calendars can be defined that way, since a calendar need not be shift-invariant.

One way to think of a calendar is that it provides a way to make continuous time feel like discrete time at a certain scale. Thus we can talk of the next year, next month, next minute etc., and combine these together to refer to the third hour of the second day of the ninth week in 1995.

A *simple calendar* is a clock with a finite sequence of durations which add up to its beat. Years divided into months and days divided into hours are examples. The beat of a simple calendar is called its *scale*., and the sequence of durations is its *rhythm* For example, our conventional year-scale calendar's rhythm is the sequence <31 days, 28days, 31days,30days,31 days,30days,31 days,31 days,30 days,31 days,30 days,31 days>. (This description insists that this is true even in a leap year, by the way: the 29th of February is always a shift-interval rather than a part of the calendar pattern.)

```
;;simplecalendar-CL
(defrelation simplecalendar
(forall (?c)
        (=> (simplecalendar ?c)
            (and (simpleclock ?c)
                 (= (beat ?c) (total (rhythm ?c))))))
)
```

Since KIF sequences are finite, such pathological examples as an oscillator approaching infinite frequency are ruled out. A rhythm defines a sequence of intervals between successive clockticks.

```
;;dates-CL
(deffunction dates
(forall (?c ?p) (=> (and (simplecalendar ?c) (clocktick ?p ?c))
                    (and (= (beginof (first (dates ?c ?p)) ?p)
```

```
                (meets (dates ?c ?p))
                (= (rhythm ?c) (dduration (dates ?c ?p))  )
                )
            )))
```

So the dates of a year are the months, and the dates of a day are the hours. (This axiom should be rewritten for the point continuum, since the simple requirement of meeting may be too simplistic. It would lead for example to some hours being open and some closed. The proper thing to say there is that there is a moment between them.)

6.5 Correcting and adjusting Clocks

Clocks are prone to corrections of various kinds. A clock may be adjusted to be faster or slower, and clocks can be given deliberate hiccups as in leap years. Since leap years are regular, one could define a four-year clock to take them into account, but an alternative approach is to introduce the notion of a correction. Since our definition of clock insists that they never change, we have to describe making a correction as shifting to a different, but usually closely related, clock. For example, leap years and leap seconds re-set the clock's `starttime` slightly:

```
;;shift-CL
(defrelation shift (?c ?d ?e) :=
(and (= (beat ?c) (beat ?d))
     (= ?e (duration (between (starttime ?c) (starttime ?d))))))
```

The shifting interval need not be measurable by the clock. It can be much smaller than the beat of the clock, as in usually the case with such minor corrections.

Correcting the rate just amounts to re-setting the beat; but as this happens at a particular time, we usually implicitly consider the start-time also to have shifted to keep things straight.

```
;;adjust-CL
(defrelation adjust (?c ?d ?e ?p) :=
(and  (= (beat ?d) (add (beat ?c) ?e))
      (= (simpleclocktime ?p ?c)(simpleclocktime ?p ?d))
      )
)
```

It follows that when a `clock` is adjusted, its `starttime` moves to the timepoint which it would have to have been in order to have arrived at this time at the new rate!

Adjust makes a clock slower. The usefulness of allowing negative durations and backward intervals is shown by the fact that to make a clock faster is simply to make it slower by a negative amount.

We can now consider an intuitive 'clock' to be a series of simple clocks, each one differing from the previous one by a correction of some kind happening at a point. In the point continuum, the changes occurring during single-point closed intervals. In the glass continuum, the changes happen at the moments where the longer intervals meet each other. Either way, time would be ambiguous only for a moment.

Since this correcting might go on for ever, the sequence of clocks might be infinite, so cannot be described as a KIF sequence. It is simplest to define a function. Every clock has an associated set of points, called the `resettimes`, totally ordered by `before`. If this is empty then the clock is simple clock. We need the idea of the next reset time, provided by:

```
;;nextone-CL
(deffunction nextone
  (and  (member (nextone ?p ?s) ?s)
        (before ?p (nextone ?p ?s))
        (forall (?x) (=> (member ?x ?s)
                         (not (before ?p ?x (nextone ?p ?s)))))
  ))
```

I will now slip into second-order syntax for a moment, in order to define this predicate on functions:

```
;;clock-CL-1
(forall (?c)
        (<=> (clock ?c)
             (and (forall (?x) (=> (member ?x (resettimes ?c))
                                   (timepoint ?x) ))
                  (forall (?p)
                          (=> (member ?p (resettimes ?c))
                              (simpleclock (value ?c ?p))))
                   )))
```

and then of course the time as measured by a clock is the clocktime on the simple clock currently in use:

```
;;clock-CL-2
(forall (?p ?c)
        (or
         (and (empty (resettimes ?c))
              (simpleclock ?c)
              (= (clocktime ?p ?c) (simpleclocktime ?p ?c))  )
         (forall (?q )
                 (<=>
                  (and (member ?q (resettimes ?c))
                       (before ?q ?p (nextone ?q
                                             (resettimes ?c))))
                  (= (clocktime ?p ?c)
                     (simpleclocktime ?p (value ?c ?q)) )))))
```

6.6 Compound clocks and calendars

Days are a particularly simple form of calendar in which the inner beats are themselves regular clockticks. It is worth giving this class a special name, since everything becomes so much simpler in this case. Let us say that one simple clock `fits` another if all its clockticks are also clockticks of the second:

```
;;fits-cl
(deffunction fits)
(forall (?c ?d )
```

```
        (<=> (fits ?c ?d)
             (forall (?p) (=> (clocktick ?p ?c) (clocktick ?p ?d)))))
```

Then it follows that the first one's beat is an integral number of beats of the second. A sequence that long of identical beats of the second clock would serve as the rhythm of a calendar whose beat was the first clock. This is exactly the relationship between minutes and seconds, days and hours, and years and weeks. That these clocks all fit together so nicely is the reason why any time can be defined as a certain number of milliseconds after the birth of Christ. In discrete time, every clock fits onto the quantum clock, but in dense time arbitrarily fine misfits of rate can occur.

We can now define a compound clock to be a sequence of clocks, each of which fits the next , which restarts at the beginning of each beat of the first. The basic relation is between a simple clock and a (not simple) clock which gets reset at midnight:

```
;;dividesup-CL
(defrelation dividesup
(forall (?c ?d)
        (<=> (dividesup ?c ?d)
             (and (forall (?p)
                          (and (<=> (clocktick ?p ?c)
                                    (and (member ?p (resettimes ?d))
                                         (= ?p
                                            (starttime (value ?d ?p)))
                                         )
                                    )
                               )
                          ))))
)
```

and then:

6.6.3
```
;;compoundclock-CL
(defrelation compoundclock)
(forall (?c ?d @s)
        (<=> (compoundclock ?c ?d @s)
             (and  (dividesup (value ?c ?d))
                   (compoundclock ?d @s))))
```

So each of the slower clocks has to inherit the same corrections, if any, that are made to the faster ones. This means for example that adding a leap second at midnight on the 17th of July makes the enclosing week, year or century also shift forward by that second, if the whole nested structure is asserted to be a compound clock.

There is almost no end to the temporal structures that could be defined and that might be useful. For example, this document has not considered intermittent intervals, or developed the idea of branching-time clocks, or transitivity tables for relativistic time. However, these ideas are clearly capable of considerable expressive power, and I hope that the careful comparative development might be of some utility to future temporal formalizers.

7. Acknowledgements

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